

Local Solvability in L^p of First-Order Linear Operators

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We prove that first-order partial differential operators of principal type with smooth coefficients are locally solvable in L^p , $1 < p < \infty$, if they satisfy condition (\mathcal{P}). © 1996 Academic Press, Inc.

1. INTRODUCTION

Consider a linear partial differential operator with smooth coefficients in an open subset Ω of \mathbb{R}^n

$$P(x, D)u = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u, \quad u \in C_c^\infty(\Omega), \quad (1.1)$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$ denotes a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_n$ its length, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ and $D_j = \partial/\partial x_j$. The principal symbol of $P(x, D)$ is the function

$$p_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n.$$

Here, $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$. If we interpret $p_m(x, \xi)$ as defined on the cotangent bundle $T^*(\Omega)$, the principal symbol becomes invariantly defined under change of coordinates. A linear partial differential operator is of principal type if

$$p_m(x, \xi) = 0, \quad \xi \in \mathbb{R}^n \setminus 0 \quad \text{implies} \quad \nabla p_m(x, \xi) \neq 0.$$

* Partly supported by CNPq.

† Partly supported by CAPES.

We recall

DEFINITION 1.1. *A partial differential operator $P(x, D)$ in $\Omega \subset \mathbb{R}^n$ is locally solvable if every point $x_0 \in \Omega$ has a neighborhood $U \subset \Omega$ such that the equation*

$$P(x, D)u = f \quad (1.2)$$

can be solved in $\mathcal{D}'(U)$ for every $f \in C_c^\infty(U)$.

Nirenberg and Treves [18, 19] introduced their condition (\mathcal{P}) bearing on the principal symbol of the operator under study and proved that under certain hypotheses it was equivalent to local solvability. Then, Beals and Fefferman [3] proved that (\mathcal{P}) implies local solvability in L^2 with loss of one derivative for a general operator of principal type, in the sense of the following definition:

DEFINITION 1.2. *A partial differential operator of order m $P(x, D)$ defined in $\Omega \subset \mathbb{R}^n$ is locally solvable in L^p if for every point $x_0 \in \Omega$ and every $s \in \mathbb{R}$ there is a neighborhood $U \subset \Omega$ of x_0 , such that for every $f \in L_s^p \cap \mathcal{E}(U)$ the equation*

$$P(x, D)u = f \quad (1.3)$$

can be solved in U with $u \in L_{s+m-1}^p$. Here, $L_s^p = (I - \Delta)^{-s/2} L^p(\mathbb{R}^n)$.

Since (\mathcal{P}) is a necessary condition even for weaker forms of local solvability [17, 9] the question of local L^p solvability is settled for $p = 2$. It is a remarkable fact that the situation is quite different for $p \neq 2$. A standard duality argument shows that the local solvability in L^p of the operator P implies a local *a priori* estimate of the form $\|u\|_{p', m-1} \leq C\|{}^t P u\|_{p'}$, for all test functions u supported in some neighborhood, where ${}^t P$ indicates the formal transpose of P . On the other hand, Littman [16] showed that an *a priori* estimate

$$\|u\|_{p, 1} \leq C\|Du\|_p, \quad u \in C_c^\infty(\mathbb{R}^n)$$

cannot hold as long as $p > 2n/(n-1)$, if D denotes the wave operator (which is of principal type and—having constant coefficients—satisfies (\mathcal{P})). Then, Kenig and Tomas [13, 14] showed that the *a priori* estimate

$$\|u\|_p \leq C\|(D + i)u\|_p, \quad u \in C_c^\infty(\mathbb{R}^3)$$

holds if and only if $p = 2$. For local versions of these negative results see [12] and the references given therein. More recently, P. Guan [7] considered the operator on $\mathbb{R}_t \times \mathbb{R}^n$

$$P = \frac{\partial^2}{\partial t^2} - \Delta_x - it^{2k}\Delta_x, \quad (1.4)$$

where Δ_x is the Laplace operator acting on \mathbb{R}_x^n and k is a positive integer. Then, given $p > 2$, one can choose n large enough, so that P given by (1.4) is not solvable in L^p .

On the other hand, in this paper we show that for first-order operators with smooth coefficients condition (\mathcal{P}) is enough to guarantee local solvability in L^p for any $1 < p < \infty$. More precisely, consider the differential operator of order one with smooth complex coefficients

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^n a_j(x, t) \frac{\partial}{\partial x_j} + c(x, t)$$

defined in a neighborhood Ω of the origin of \mathbb{R}^{n+1} . The principal symbol l of L is defined on the cotangent bundle $T^*(\mathbb{R}^{n+1})$ by the identity $l(d\phi) = L(\phi)$, $\phi \in C^\infty(\Omega; \mathbb{R})$. Then, L is said to satisfy (\mathcal{P}) in an open set $\Omega \subset \mathbb{R}^{n+1}$ if there is no complex-valued function $g \in C^\infty(\Omega)$ such that $\text{Im}(gl)$ takes both positive and negative values on a null bicharacteristic of $\text{Re}(gl)$, where $g \neq 0$. We recall that a bicharacteristic of a real function f defined on $T^*(\Omega)$ is an integral curve of the Hamiltonian field H_f . Since $H_f f = 0$, f is constant along its bicharacteristics; when the constant is zero the bicharacteristic is said to be null. Note that the definition of (\mathcal{P}) is coordinate free. After a local change of variables, the operator can be put in the form

$$L = \frac{\partial}{\partial t} + i \sum_{j=1}^n b_j(x, t) \frac{\partial}{\partial x_j} + c(x, t) \quad (1.5)$$

with $b_j(x, t)$ smooth and real.

THEOREM 1.1. *Assume that L given by (1.5) satisfies condition (\mathcal{P}) . Then L is locally solvable in L^p , for $1 < p < \infty$.*

The paper is organized as follows: In Section 2, we prove the necessary *a priori* estimates for a special situation in two variables. In Section 3, we recall how the general case is reduced by localization to the special case already considered and in Section 4 Theorem 1.1 is proved.

2. THE BASIC ESTIMATE

Consider the first-order linear differential operator in two variables

$$L = \frac{\partial}{\partial t} - ib(x, t) \frac{\partial}{\partial x}, \quad x \in \mathbb{R}, \quad |t| < T. \quad (2.1)$$

We write $\Omega_T = \mathbb{R} \times [-T, T]$ and assume that

- (i) $b(x, t)$ is smooth, real and nonnegative,
- (ii) all derivatives of $b(x, t)$ are uniformly bounded.

The size of T will be decreased a number of times. We also write

$$B(x, t, t') = \int_{t'}^t b(x, s) ds.$$

The next lemma describes a function which is central to the construction of a parametrix for L . This parametrix (with minor modifications) has been used to study the global hypoellipticity of L in [10], and also in [2] in connection with solvability of a class of semilinear equations.

LEMMA 2.1. *Let L be as above. There exists a function $\phi(x, t, t')$ in $\Omega_T \times [-T, T]$, such that $x - \phi(x, t, t')$ is bounded with bounded derivatives and such that*

$$|D_x^\alpha D_t^\beta D_{t'}^\gamma (L\phi(x, t, t'))| \leq C(N, \alpha, \beta, \gamma) |B(x, t, t')|^N, \quad N = 0, 1, \dots \quad (2.2)$$

and

$$\phi(x, t', t') = x, \quad (x, t') \in \Omega_T. \quad (2.3)$$

Furthermore, if T is decreased conveniently we also obtain that

$$\begin{aligned} \frac{1}{2}B(x, t, t') &\leq \operatorname{Im} \phi(x, t, t') \leq \frac{3}{2}B(x, t, t'), & t \geq t', \\ \frac{3}{2}B(x, t, t') &\leq \operatorname{Im} \phi(x, t, t') \leq \frac{1}{2}B(x, t, t'), & t \leq t', \end{aligned} \quad (2.4)$$

and

$$|\operatorname{Re} \phi_x(x, t, t') - 1| < 1/2. \quad (2.5)$$

We need to consider pseudo-differential operators in the class $\mathcal{L}_{\rho, \delta}^m$ (cf. [1, 8, 15]), $m \in \mathbb{R}$, $0 < \rho \leq 1$, $0 \leq \delta < 1$; actually, only the case $\rho = 1$, $\delta = 1/2$ is of interest. These are operators L of the form

$$Lf(x) = (2\pi)^{-n} \int e^{ix \cdot \xi} p(x, \xi) \hat{f}(\xi) d\xi, \quad f \in \mathcal{S}, \quad (2.6)$$

when n is the dimension of the euclidean space. The function $p(x, \xi)$, uniquely determined by L and called the symbol of L , is assumed to belong to the class $S_{\rho, \delta}^m$. This means that it is a smooth function satisfying the estimates

$$|D_x^\alpha D_\xi^\beta p(x, \xi)| \leq C_{\alpha\beta} (1 + |\xi|)^{m - \rho|\beta| + \delta|\alpha|}. \quad (2.7)$$

We recall [1, 5] the following continuity result:

THEOREM 2.2. *Let $L \in \mathcal{L}_{\rho, \delta}^m$, $m \in \mathbb{R}$, $0 < \rho \leq 1$, $0 \leq \delta < 1$. Then $L \in \mathcal{L}(L^p)$, provided that, $m \leq m_p = -n(1/p - 1/2)(1 - \rho) + \lambda$, $\lambda = \max(0, (\delta - \rho)/2)$, $1 < p < \infty$.*

Now we consider a function $0 \leq \eta^+(\xi) \leq 1 \in C^\infty(\mathbb{R})$ such that $\eta^+(\xi) = 0$ if $\xi \leq -1$ and $\eta^+(\xi) = 1$ if $\xi \geq 1$ and set $\eta^- = 1 - \eta^+$.

LEMMA 2.2..

(i) For $-T \leq t' \leq t \leq T$ the function

$$a^+(x, \xi, t, t') = \eta^+(\xi) \exp(-\operatorname{Im} \phi(x, t, t') \xi)$$

is a symbol of class $S_{1,1/2}^0(\mathbb{R})$ as a function of (x, ξ) depending continuously on the parameters t, t' . More generally,

$$D_t^j D_{t'}^k a^+(x, \xi, t, t') \in S_{1,1/2}^{j+k}(\mathbb{R}), \quad j, k = 0, 1, \dots$$

uniformly and continuously on t, t' for $t' \leq t$.

(ii) Similarly,

$$a^-(x, \xi, t, t') = \eta^-(\xi) \exp(-\operatorname{Im} \phi(x, t, t') \xi)$$

satisfies, as a function of (x, ξ) ,

$$D_t^j D_{t'}^k a^-(x, \xi, t, t') \in S_{1,1/2}^{j+k}(\mathbb{R}), \quad j, k = 0, 1, \dots$$

uniformly and continuously on $t \leq t'$.

Proof. It is enough to prove (i), for the proof of (ii) is analogous. Clearly, (2.4) shows that $\operatorname{Im} \phi(x, t, t') \geq 0$ for $t' \leq t$ and the reverse inequality holds for $t \leq t'$. Thus, $|a^+(x, \xi, t, t')| \leq C$ because ϕ is bounded. Consider the function $\sqrt{\operatorname{Im} \phi(x, t, t')}$ for $t' \leq t$. By a result of Glaeser [6, 4] it is continuously differentiable and any first-order derivative is uniformly bounded (recall that all derivatives of ϕ are bounded). This gives the estimate

$$|D_x \operatorname{Im} \phi(x, t, t')| \leq C \sqrt{\operatorname{Im} \phi(x, t, t')},$$

which implies for $t' \leq t$

$$|D_x a^+(x, \xi, t, t')| \leq C(1 + |\xi|)^{1/2}$$

using the trivial estimate $\sqrt{s} e^{-s} \leq C$, $s \geq 0$. Similarly, the estimate $se^{-s} \leq C$, $s \geq 0$, yields

$$|D_\xi a^+(x, \xi, t, t')| \leq C(1 + |\xi|)^{-1}$$

and by induction one gets $|D_x^j D_\xi^k a^+| \leq C_{jk}(1 + |\xi|)^{j/2-k}$ for $j, k = 0, 1, \dots$ and $t' \leq t$. The estimates for the derivatives of a^+ with respect to t and t' can be treated in the same way.

Now set for $f \in C_c^\infty(\Omega_T)$

$$K^+ f(x, t) = \frac{1}{2\pi} \int_{-T}^t \int_{-\infty}^{\infty} e^{i\phi(x, t, t')\xi} \eta^+(\xi) \hat{f}(\xi, t') d\xi dt',$$

$$K^- f(x, t) = \frac{1}{2\pi} \int_T^t \int_{-\infty}^{\infty} e^{i\phi(x, t, t')\xi} \eta^-(\xi) \hat{f}(\xi, t') d\xi dt',$$

where $\hat{f}(\xi, t')$ indicates the partial Fourier transform of the function $f(x, t')$ with respect to the first variable. If we write

$$R^+ f(x, t) = \frac{1}{2\pi} \int_{-T}^t \int_{-\infty}^{\infty} e^{i\phi(x, t, t')\xi} i\xi L\phi(x, t, t') \eta^+(\xi) \hat{f}(\xi, t') d\xi dt',$$

$$R^- f(x, t) = \frac{1}{2\pi} \int_T^t \int_{-\infty}^{\infty} e^{i\phi(x, t, t')\xi} i\xi L\phi(x, t, t') \eta^-(\xi) \hat{f}(\xi, t') d\xi dt',$$

it follows from direct computation that

$$LK^+ f = \eta^+(D)f + R^+ f,$$

$$LK^- f = \eta^-(D)f + R^- f.$$

Observe that, in view of (2.2), we have for any $N = 0, 1, \dots$ and $t' \leq t$

$$\begin{aligned} & |\exp(-\operatorname{Im} \phi(x, t, t')\xi) L\phi(x, t, t') \eta^+(\xi)| \\ & \leq C_N \exp(-B(x, t, t')\xi/2) B(x, t, t')^N \eta^+(\xi) \\ & \leq C_N (1 + |\xi|)^{-N}, \end{aligned}$$

and similar estimates hold for $D_x^j D_t^k D_{t'}^l \exp(-\operatorname{Im} \phi\xi) L\phi(x, t, t') \eta^-(\xi)$. Hence, we may regard R^+ (resp. R^-) as a smooth function of t and t' with values in the space of regularizing operators $\mathcal{L}^{-\infty}(\mathbb{R})$ that map $\mathcal{S}'(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$. Writing $K = K^+ + K^-$ and $R = R^+ + R^-$ we obtain

$$LKf = f + Rf \tag{2.8}$$

since by construction $\eta^+(D) + \eta^-(D) = I$.

For fixed t and t' the function $x \rightarrow \operatorname{Re} \phi(x, t, t')$ is a diffeomorphism of \mathbb{R} with bounded derivatives by Lemma 2.1. Let $\psi(x, t, t')$ denote its inverse. It follows that the composite

$$\tilde{a}^+(x, \xi, t, t') = a^+(\psi(x, t, t'), \xi, t, t')$$

is also a symbol in $S_{1/2,1}^0$ with the same properties as a^+ . If $A_{t,t'}^+$ denotes the pseudo-differential operator with symbol $\tilde{a}^+(x, \xi, t, t')$ depending on t, t' as parameters, then

$$K^+f(x, t) = \int_{-T}^t (A_{t,t'}^+f) \circ \operatorname{Re} \phi \, dt',$$

with an analogous formula for K^- . It follows that the operators $A_{t,t'}^+$ and $A_{t,t'}^-$ are bounded in L^p , $1 < p < \infty$, and their operator norms have a bound independent of $(t, t') \in [-T, T]^2$, if p is kept fixed. We get right away that for $f \in C_c^\infty(\Omega_T)$

$$\|K^+f(\cdot, t)\|_p^p \leq M^p \left(\int_T^t \|f(\cdot, t')\|_p \, dt' \right)^p \leq M^p (2T)^{p/p'} \|f\|_p^p,$$

where $1/p + 1/p' = 1$. Integrating now in t we get

$$\|K^+f\|_p \leq 2MT\|f\|_p.$$

Since a similar estimate holds for K^-f we obtain

$$\|Kf\|_p \leq CT\|f\|_p, \quad f \in C_c^\infty(\Omega_T). \quad (2.9)$$

Similarly, we get

$$\|Rf\|_p \leq CT\|f\|_p, \quad f \in C_c^\infty(\Omega_T). \quad (2.10)$$

Consider now $f, g \in C_c^\infty(\Omega_T)$ and rewrite (2.8) as

$$\langle Kf, L^t g \rangle = \langle f, g \rangle + \langle Rf, g \rangle,$$

where $\langle f, g \rangle = \int fg \, dx \, dt$ and L^t denotes the transpose of L . This implies trivially

$$|\langle f, g \rangle| \leq |\langle Kf, L^t g \rangle| + |\langle Rf, g \rangle|.$$

Let $1/p + 1/q = 1$; consider, in the above inequality, the supremum over all $f \in C_c^\infty(\Omega_T)$ such that $\|f\|_p = 1$. In view of (2.9) and (2.10) we get

$$\|g\|_q \leq CT\|L^t g\|_q + CT\|g\|_q, \quad g \in C_c^\infty(\Omega_T),$$

which implies, for T small enough and a larger C ,

$$\|g\|_q \leq CT\|Lg\|_q, \quad g \in C_c^\infty(\Omega_T). \quad (2.11)$$

In the last step we have used that $L^t = -L + ib_x$ and absorbed the error term $CT(\|g\|_q + \|b_x\|_\infty\|g\|_q)$. Finally, interchanging the role of q and p we may write (2.11) as

$$\|g\|_p \leq CT\|Lg\|_p, \quad g \in C_c^\infty(\Omega_T). \quad (2.12)$$

We observe that the constant C in (2.12) only depends on the size of the coefficient $b(x, t)$ of L and of its derivatives up to a certain fixed order that depends only on p . ■

3. PATCHING THE BASIC ESTIMATES

We turn to the operator defined on $\mathbb{R}_x^n \times \mathbb{R}_t$

$$L = \frac{\partial}{\partial t} - i \sum_{j=1}^n b_j(x, t) \frac{\partial}{\partial x_j} + c(x, t), \quad (3.1)$$

where the coefficients $b_j(x, t)$ are real smooth functions and $c(x, t)$ is smooth and complex. Since we are interested in a local result we may further assume that the coefficients vanish identically outside a compact set.

THEOREM 3.1. *Let $1 < p < \infty$. If the operator (3.1) satisfies condition (\mathcal{P}) there exist $T_0 > 0$ and $C > 0$ such that for every $0 < T < T_0$*

$$\|u\|_p \leq CT \|Lu\|_p, \quad u \in \mathcal{C}_c^\infty(\mathbb{R}^n \times (-T, T)). \quad (3.2)$$

We fix a value of $1 < p < \infty$. It is possible to pass from the L^p estimates (2.12) in two variables to the general estimate (3.2) exactly in the same way this is done when $p = 2$ [11, 20]. We recall briefly the main steps for the sake of completeness. First, we observe that it is enough to prove (3.2) when $c(x, t) \equiv 0$, because the contribution of this term is bounded by $CT\|u\|_p$, which can be absorbed for T small.

We denote $\mathbf{b}(x, t)$ the vector field in \mathbb{R}^n given by $\sum_{j=1}^n b_j(x, t) \partial / \partial x_j$. The fact that L satisfies (\mathcal{P}) means that there exists a unit vector field $\mathbf{v}(x)$ defined on \mathbb{R}^n such that

$$\mathbf{b}(x, t) = |\mathbf{b}(x, t)| \mathbf{v}(x), \quad x \in \mathbb{R}_x^n, t \in \mathbb{R}.$$

Set

$$\mathcal{N} = \{x \in \mathbb{R}^n : \mathbf{b}(x, t) = 0, |t| < 1\}$$

and

$$\rho(x) = \sup_{|t| < 1} |\mathbf{b}(x, t)|, \quad x \in \mathbb{R}^n,$$

so that \mathcal{N} is precisely the set where ρ vanishes. The function $\rho(x)$ is Lipschitz and $\|\nabla \rho\|_{L^\infty} \leq \|\nabla_x \mathbf{b}\|_{L^\infty}$. We point out that, outside \mathcal{N} , the vector $\mathbf{v}(x)$ is smooth and satisfies the inequality $|\nabla \mathbf{v}(x)| \leq 2\|\nabla \mathbf{b}\|_{L^\infty} / \rho(x)$ for

$x \notin \mathcal{N}$. Assuming, without loss of generality, that $\|\nabla \mathbf{b}\|_{L^\infty} \leq 1$, the following estimates hold

$$\begin{aligned} |\rho(x) - \rho(x')| &\leq |x - x'|, & x, x' \in \mathbb{R}^n, \\ |\nabla \mathbf{v}(x)| &\leq \frac{2}{\rho(x)}, & x \notin \mathcal{N}. \end{aligned} \quad (3.3)$$

PROPOSITION 3.2. *Let χ be the characteristic function of \mathcal{N} and let L_0 denote the principal part of L . Then*

$$L_0 \chi = 0$$

in the sense of distributions.

The proposition follows easily from Fubini's theorem and the next

LEMMA 3.3. *Let $b(x)$, $x \in \mathbb{R}$, be a Lipschitz function, let G be a measurable subset of the set where $b(x)$ vanishes, and let χ be the characteristic function of G . Then $b d\chi/dx = 0$ in the sense of distributions.*

Proof. We must show that

$$\int_G (b'u + bu') dx = 0, \quad u \in C_c^\infty.$$

Since b vanishes on G , only the term $b'u$ matters. But the set of points $x \in G$ where $b'(x)$ is defined and not zero is discrete, hence of measure zero, while the set of points where $b'(x)$ is not defined is also negligible.

In view of Proposition 3.2, $[L, \chi] = 0$ so to obtain (3.2) it is enough to prove separately the inequalities

$$\|\chi u\|_p \leq CT \|L\chi u\|_p, \quad u \in \mathcal{E}_c^\infty(\mathbb{R}^n \times (-T, T)), \quad (3.4)$$

$$\|(1 - \chi)u\|_p \leq CT \|L(1 - \chi)u\|_p, \quad u \in \mathcal{E}_c^\infty(\mathbb{R}^n \times (-T, T)). \quad (3.5)$$

Now $L\chi u = \chi LU = \chi u_t$ so (3.4) is easy to obtain. The proof of (3.5) uses the following variation of a lemma of Whitney. ■

LEMMA 3.4. *Let $\mu > 0$. There exists an open covering of $\mathbb{R}^n \setminus \mathcal{N}$ by cubes with sides parallel to the coordinate axes $\{Q_k\}$, $k = 1, 2, \dots$, such that the intersection of N cubes of the family is always empty, for certain fixed integer N , and we have the estimates*

$$\text{diam } Q_k \leq \mu \inf_{Q_k} \rho(x) \leq \mu \sup_{Q_k} \rho(x) \leq 3 \text{diam } Q_k, \quad k = 1, 2, \dots \quad (3.6)$$

Furthermore, there are functions $\phi_k \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus \mathcal{N})$ such that $\{\phi_k^p\}$ is a partition of unity in $\mathbb{R}^n \setminus \mathcal{N}$ subordinated to the covering $\{Q_k\}$ and for a certain constant $C > 0$,

$$\|\nabla \phi_k\|_{L^\infty} \leq \frac{C}{\text{diam } Q_k}, \quad k = 1, 2, \dots \quad (3.7)$$

Assume that u is supported in $Q_k \times (-T, T)$ for a certain cube of the covering. Then (3.3) and (3.6) show that $\mathbf{v}(x)$ is approximately constant on Q_k if μ is small and we can rectify its flow. Assume, without loss of generality, that $v_1(x) \geq (1/2\sqrt{n})$ on Q_k . Solving the differential equations

$$\frac{dx_j}{dy_1} = \frac{v_j(x)}{v_1(x)}, \quad x_j(0) = y_j, \quad j = 2, \dots, n \quad (3.8)$$

we obtain a smooth change of variables on Q_k given by $x_1 = y_1$, $x_j = x_j(y_1; y_2, \dots, y_n)$, where the right-hand side denotes the solution of (3.8). In the new coordinates $\mathbf{v}(x(y)) = v_1(x(y))\partial/\partial y_1$ and L assumes the form

$$\frac{\partial}{\partial t} - ib_1(x(y), t) \frac{\partial}{\partial y_1} \quad (3.9)$$

with $b_1 > 0$.

For small T , we now invoke estimate (2.12) to get $\|u\|_p \leq CT\|Lu\|_p$ for functions $u \in \mathcal{C}_c^\infty(Q_k \times (-T, T))$. This estimate is, in principle, valid for the L^p norm in the y coordinates but, as the Jacobian of the change of variables is bounded by a fixed constant, it is also true in the x variables. In particular, we obtain

$$\|\phi_k u\|_p \leq CT\|L\phi_k u\|_p, \quad u \in \mathcal{C}_c^\infty(\mathbb{R}^n \times (-T, T)). \quad (3.10)$$

It is important to note that both T and C can be taken to be independent of k . Raising both sides of (3.10) to the exponent p and adding the resulting inequalities for $k = 1, 2, \dots$ we obtain

$$\begin{aligned} \|(1 - \chi(x))u\|_p^p &\leq C^p T^p \sum_{k=1}^{\infty} \|L\phi_k u\|_p^p \\ &\leq C_p^p T^p \|(1 - \chi(x))Lu\|_p^p \\ &\quad + C_p^p T^p \int \sum_{k=1}^{\infty} |[L, \phi_k]|^p |(1 - \chi(x))u|^p dx dt. \end{aligned} \quad (3.11)$$

It follows from (3.6) and (3.7) that the second term on the right-hand side of (3.11) can be controlled by CT^p . This is so because $|\nabla\phi_k|$ is bounded by $\text{diam}(Q_k)^{-1}$ so that we have the pointwise estimate $\|L, \phi_k\| = |\mathbf{b} \cdot \nabla\phi_k| \leq \sup_{Q_k} \rho \times \sup |\nabla\phi_k| \leq C$ which combined with (3.11) gives for $T > 0$ sufficiently small,

$$\|(1 - \chi(x))u\|_p \leq CT\|(1 - \chi(x))Lu\|_p, \quad u \in \mathcal{E}_c^\infty(\mathbb{R}^n \times (-T, T)).$$

This proves (3.5).

4. PROOF OF THEOREM 1.1

It is well known that, starting from inequality (3.2) and considering the commutators of L with the operators $J_s = (I - \Delta)^{s/2}$, one may derive for any real s and any positive T the inequality

$$\|u\|_{p,s} \leq CT\|Lu\|_{p,s} + M\|u\|_{p,s-1}, \quad u \in \mathcal{E}_c^\infty(U), \quad (4.1)$$

where U is some neighborhood of the origin and M is a positive constant, both depending on T , p , and s . Then, general arguments [19, p. 468] allow us to obtain the estimate

$$\|u\|_{p,s} \leq C\|Lu\|_{p,s}, \quad u \in \mathcal{E}_c^\infty(U), \quad (4.2)$$

for convenient $C > 0$ and $0 \in U \subset \mathbb{R}^{n+1}$. Finally, (4.2) and the Hanh–Banach theorem can be used to solve the transposed equation $L'u = f$ in U , for $f \in L_{-s}^{p'} \cap \mathcal{E}'(U)$, $1/p + 1/p' = 1$, with $u \in L_{-s}^{p'}$. Since L satisfies condition (\mathcal{P}) if and only if L' does, this proves Theorem 1.1.

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